

Submitted to: Annals of Mathematics

## GENERALIZED JACOBIAN FOR FUNCTIONS WITH INFINITE DIMENSIONAL RANGE AND DOMAIN

ZSOLT PÁLES AND VERA ZEIDAN

**ABSTRACT.** In this paper, locally Lipschitz functions acting between infinite dimensional normed spaces are considered. When the range is a dual space and satisfies the Radon–Nikodým property, Clarke’s generalized Jacobian will be extended to this setting. Characterization and fundamental properties of the extended generalized Jacobian are established including the nonemptiness, the  $\beta$ -compactness, the  $\beta$ -upper semicontinuity, and a mean-value theorem. A connection with known notions is provided and chain rules are proved using key results developed. This included the vectorization and restriction theorem, and the extension theorem. Therefore, the generalized Jacobian introduced in this paper is proved to enjoy all the properties required of a derivative-like-set.

### 1. Introduction

The subject of nonsmooth analysis focuses on the study of a derivative-like object for nonsmooth functions. When the function is a convex real-valued, the notion of subgradient was introduced in the late fifties by Rockafellar in [36], and in the references therein. Since then, the focus shifted to finding derivative-like objects for nonconvex, in particular, locally Lipschitz functions acting between two normed spaces  $X$  and  $Y$ .

When  $X$  and  $Y$  are *both* finite dimensional normed spaces and  $f : \mathcal{D} \rightarrow Y$  is a *vector-valued* locally Lipschitz function, Clarke introduced in [6], [7] the notion of the *generalized Jacobian* based on Rademacher’s theorem which gives the almost everywhere differentiability of locally Lipschitz functions. This generalized Jacobian is

$$\partial f(p) := \text{co} \left\{ A \in \mathcal{L}(X, Y) \mid \exists (x_i)_{i \in \mathbb{N}} \text{ in } \Omega(f) : \lim_{i \rightarrow \infty} x_i = p \text{ and } \lim_{i \rightarrow \infty} Df(x_i) = A \right\}, \quad (1.1)$$

where  $\Omega(f)$  denotes the set of the points of  $\mathcal{D}$  where  $f$  is differentiable, it is of full Lebesgue measure.

Another but related generalized Jacobian based also on the use of Rademacher’s theorem was proposed for the same setting by Pourciau in [31] and was defined as

$$\partial^P f(p) := \bigcap_{\delta > 0} \overline{\text{co}} \left\{ Df(x) : x \in (p + \delta B_X) \cap \Omega(f) \right\}. \quad (1.2)$$

One can show that Clarke’s and Pourciau’s generalized Jacobians are equivalent. These two objects are nonempty, due to Rademacher’s theorem. Furthermore, in terms of these Jacobians, results have been derived pertaining optimality conditions, implicit functions theorems, metric regularity, and calculus rules including the sum rule and the chain rule. Thereby, it has already been shown that these generalized Jacobians are successful approximations of  $f$  by linear operators.

---

*Date:* February 2, 2008.

*2000 Mathematics Subject Classification.* 49J52, 49A52, 58C20.

*Key words and phrases.* Generalized Jacobian, Characterization theorem, Restriction Rule, Chain rule, Sum rule, Continuous selection.

Research of the first author is supported by the Hungarian Scientific Research Fund (OKTA) under grant K62316. Research of the second author is supported by the National Science Foundation under grant DMS-0306260.

When establishing calculus rules such as the sum and/or the chain rules, a fundamental property, namely, the “blindness” of the generalized Jacobian with respect to sets of Lebesgue measure zero, or null sets is needed. The blindness of Clarke generalized gradient was established by Clarke in [7] and that of Clarke’s generalized Jacobian was shown by Warga [43] and by Fabian & Preiss [9].

Thibault in [41] extended Clarke’s notion of generalized Jacobian, equation (1.1), to the case where  $X$  and  $Y$  are infinite dimensional separable Banach spaces such that  $Y$  is *reflexive*. This extension was based on the dense differentiability of locally Lipschitz functions (cf. Aronszajn [1], Christensen [5], and Phelps [30]). Thibault’s definition is

$$\partial_H f(p) := \overline{\text{co}} \left\{ A \in \mathcal{L}(X, Y) \mid \exists (x_i)_{i \in \mathbb{N}} \text{ in } H : \lim_{i \rightarrow \infty} x_i = p \text{ and } \lim_{i \rightarrow \infty} Df(x_i) = A \right\}, \quad (1.3)$$

where  $H$  is a subset of  $\mathcal{D}$  on which  $f$  is Gâteaux-differentiable and such that  $\mathcal{D} \setminus H$  is a Haar-null set in  $\mathcal{D}$ . The notion in (1.3) depends on the choice of the set  $H$ , and hence, unlike Clarke’s generalized Jacobian, is not known to be blind with respect to the Haar-null sets. In other words, the notion in (1.3), assigns to every locally Lipschitz function, not a single object but rather a family of generalized Jacobians that is parametrized by certain null sets. Thus, based on this approach all the chain rules derived in [41] are in terms of the Haar null set  $H$ .

Other notions are known in the infinite dimensional setting, such as the notion of derivate containers in [42], [43]; the concepts of screens and “fans” in [11], [10]; the concept of shields [39]; Ioffe’s fan derivative [16], and the notion of coderivatives developed in [25]. Most of these notions are not given in terms of relevant sets of *linear operators*. A relatively recent survey on the different subdifferentials and their properties is given in [4] where also an extended list of references could be found.

In recent papers [27] and [28] Clarke’s generalized Jacobian (1.1) was extended to the case when  $X$  was any *normed* space and  $Y$  was a finite dimensional space. In these references the generalized Jacobian was defined to be a *set of linear operators* from  $X$  to  $Y$ . When the domain is infinite dimensional and the image space is  $\mathbb{R}$ , the notion introduced in [27] and [28] coincides with Clarke’s generalized gradient which is defined as

$$\partial^c f(p) := \left\{ \zeta \in X^* \mid \langle \zeta, h \rangle \leq f^\circ(p, h), \forall h \in X \right\}, \quad (1.4)$$

where

$$f^\circ(p, h) := \limsup_{\substack{x \rightarrow p \\ t \rightarrow 0^+}} \frac{f(x + th) - f(x)}{t} \quad (1.5)$$

is Clarke’s *generalized directional derivative*.

In [27] and [28], the nonemptiness, the  $w^*$ -compactness, the convexity, and the upper semicontinuity property of the extended generalized Jacobian were derived. Furthermore, a chain rule for the composition of nonsmooth locally Lipschitz maps with finite dimensional ranges was established.

The difficulty caused by the infinite dimensionality of the domain was handled in [27] and [28] by introducing an intermediate Jacobian  $\partial_L f$  defined on finite dimensional spaces  $L$  so that Rademacher theorem remains applicable.

In this paper we are interested in extending the definition of Clarke’s generalized Jacobian to the case when in addition to the domain also the range is infinite dimensional. In this case, two extra difficulties manifest. The first is the differentiability issue related to the Rademacher theorem in infinite dimension. This issue will be handled by taking image spaces satisfying the Radon–Nikodým property. This implies that the restriction of a Lipschitz function  $f : \mathcal{D} \rightarrow Y$  to a *finite dimensional* domain is almost everywhere differentiable (cf. [2][Prop. 6.41]).

The second difficulty is pertaining finding a topology in the space of linear operators  $\mathcal{L}(X, Y)$ , where the generalized Jacobian lives, so that bounded sequences would have cluster points in this

topology. To overcome this difficulty, we also assume that the image space  $Y$  is a dual of a normed space.

The goal of this paper is to provide a generalized Jacobian for locally Lipschitz functions defined between infinite dimensional normed spaces with the range  $Y$  is a dual space and satisfies the Radon–Nikodým property. We shall show that our generalized Jacobian enjoys all the fundamental properties desired from a derivative set.

In Section 2 we introduce the  $\beta$ -topology on the space of linear operators  $L(X, Y)$ . This is a  $w^*$ -operator topology induced by the predual of  $Y$ . We prove an analog of the Banach–Alaoglu theorem as well as an extension theorem which will be crucial for the proof of the nonemptiness of the generalized Jacobian. In this section we also derive results related to various upper semi-continuity properties which will be repeatedly used in the subsequent section. In Section 3 the  $L$ -Jacobian,  $\partial_L f(p)$ , and the generalized Jacobian are defined as an extension of Pourciau’s notion, equation (1.2), to infinite dimensional spaces. We also show that our generalized Jacobian could be equivalently defined in terms of cluster points, which is a definition that corresponds to Clarke’s approach. Basic properties and a characterization of the generalized Jacobian are established. A main tool named the “restriction and the vectorization” theorem is developed which is central for deriving many results in the rest of the paper.

Relationships to Thibault’s limit set, to a Ioffe type fan derivative and to Mordukhovich coderivative are given in Section 4. A generalization of Lebourg mean-value theorem is obtained as well as that the generalized Jacobian is a  $w^*$ -Hadamard prederivative. We also characterize the cases when the generalized Jacobian is a strict norm-Gâteaux or a strict  $w^*$ -Fréchet prederivative. In Section 5 we derive two chain rules: a nonsmooth-smooth, and a nonsmooth-nonsmooth one. Their proofs evoke most of the results and properties established in the previous sections. As a consequence, a sum rule follows. Finally, in Section 6 we develop results for the generalized Jacobian of continuous selections.

## REFERENCES

- [1] N. Aronszajn, *Differentiability of Lipschitzian mappings between Banach spaces*, Studia Math. **57** (1976), no. 2, 147–190. MR 54 #13562
- [2] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000. MR 2001b:46001
- [3] J. M. Borwein and W. B. Moors, *A chain rule for essentially smooth Lipschitz functions*, SIAM J. Optim. **8** (1998), no. 2, 300–308 (electronic). MR 99g:49014
- [4] J. M. Borwein and Q. J. Zhu, *A survey of subdifferential calculus with applications*, Nonlinear Anal. **38** (1999), no. 6, Ser. A: Theory Methods, 687–773. MR 2000j:49024
- [5] J. P. R. Christensen, *Measure theoretic zero sets in infinite dimensional spaces and applications to differentiability of Lipschitz mappings*, Publ. Dép. Math. (Lyon) **10** (1973), no. 2, 29–39, Actes du Deuxième Colloque d’Analyse Fonctionnelle de Bordeaux (Univ. Bordeaux, 1973), I, pp. 29–39. MR 50 #14215
- [6] F. H. Clarke, *On the inverse function theorem*, Pacific J. Math. **64** (1976), no. 1, 97–102. MR 54 #13005
- [7] ———, *Optimization and Nonsmooth Analysis*, John Wiley & Sons Inc., New York, 1983. MR 85m:49002
- [8] ———, *Analyse non lisse et optimisation*, Cours de 3ème cycle, Ecole doctorale de mathématiques, Université Paul Sabatier, 1993–1994.
- [9] M. Fabián and D. Preiss, *On the Clarke’s generalized Jacobian*, Rend. Circ. Mat. Palermo (2) Suppl. (1987), no. 14, 305–307, Proceedings of the 14th Winter School on Abstract Analysis (Srní, 1986). MR 9d:58016
- [10] H. Halkin, *Interior mapping theorem with set-valued derivatives*, J. Analyse Math. **30** (1976), 200–207. MR 58 #6102
- [11] ———, *Mathematical programming without differentiability*, Calculus of variations and control theory (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975), Academic Press, New York, 1976, pp. 279–287. Math. Res. Center, Univ. Wisconsin, Publ. No. 36. MR 55 #8927

- [12] J.-B. Hiriart-Urruty, *Characterizations of the plenary hull of the generalized Jacobian matrix*, Math. Programming Stud. (1982), no. 17, 1–12, Nondifferential and variational techniques in optimization (Lexington, Ky., 1980). MR **84f**:90080
- [13] J.-B. Hiriart-Urruty and C. Imbert, *Les fonctions d'appui de la jacobienne généralisée de Clarke et de son enveloppe plénier*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 11, 1275–1278. MR **99k**:49035
- [14] R. B. Holmes, *Geometric Functional Analysis and its Applications*, Graduate Texts in Mathematics, vol. 24, Springer, Berlin–Heidelberg, 1975.
- [15] C. Imbert, *Support functions of the Clarke generalized Jacobian and of its plenary hull*, Nonlinear Anal. **49** (2002), no. 8, 1111–1125. MR **2003h**:49024
- [16] A. D. Ioffe, *Nonsmooth analysis: differential calculus of nondifferentiable mappings*, Trans. Amer. Math. Soc. **266** (1981), no. 1, 1–56. MR **82j**:58018
- [17] H. Th. Jongen and D. Pallaschke, *On linearization and continuous selections of functions*, Optimization **19** (1988), no. 3, 343–353. MR **90a**:58018
- [18] A. Jourani and L. Thibault, *Chain rules for coderivatives of multivalued mappings in Banach spaces*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1479–1485. MR **MR1443159** (98j:49024)
- [19] D. Klatte and B. Kummer, *Nonsmooth equations in optimization*, Nonconvex Optimization and its Applications, vol. 60, Kluwer Academic Publishers, Dordrecht, 2002, Regularity, calculus, methods and applications. MR **2003c**:49001
- [20] L. Kuntz and S. Scholtes, *Structural analysis of nonsmooth mappings, inverse functions, and metric projections*, J. Math. Anal. Appl. **188** (1994), no. 2, 346–386. MR **95h**:49022
- [21] ———, *Qualitative aspects of the local approximation of a piecewise differentiable function*, Nonlinear Anal. **25** (1995), no. 2, 197–215. MR **96e**:90042
- [22] G. Lebourg, *Generic differentiability of Lipschitzian functions*, Trans. Amer. Math. Soc. **256** (1979), 125–144. MR **80i**:58012
- [23] B. S. Mordukhovich, *Metric approximations and necessary conditions for optimality for general classes of nonsmooth extremal problems*, Dokl. Akad. Nauk SSSR **254** (1980), no. 5, 1072–1076. MR **82b**:90104
- [24] ———, *Generalized differential calculus for nonsmooth and set-valued mappings*, J. Math. Anal. Appl. **183** (1994), no. 1, 250–288. MR **95i**:49029
- [25] B. S. Mordukhovich and Y. H. Shao, *Nonconvex differential calculus for infinite-dimensional multifunctions*, Set-Valued Anal. **4** (1996), no. 3, 205–236. MR **97i**:49024
- [26] ———, *Nonsmooth sequential analysis in Asplund spaces*, Trans. Amer. Math. Soc. **348** (1996), no. 4, 1235–1280. MR **96h**:49036
- [27] Zs. Páles and V. Zeidan, *Infinite dimensional Clarke generalized Jacobian*, J. Convex Anal., accepted for publications.
- [28] ———, *Infinite dimensional generalized Jacobian: properties and calculus rules*, manuscript.
- [29] J.-S. Pang and D. Ralph, *Piecewise smoothness, local invertibility, and parametric analysis of normal maps*, Math. Oper. Res. **21** (1996), no. 2, 401–426. MR **97d**:90100
- [30] R. R. Phelps, *Gaussian null sets and differentiability of Lipschitz map on Banach spaces*, Pacific J. Math. **77** (1978), no. 2, 523–531. MR **80m**:46040
- [31] B. H. Pourciau, *Analysis and optimization of Lipschitz continuous mappings*, J. Optimization Theory Appl. **22** (1977), no. 3, 311–351. MR **56** #13059
- [32] ———, *Univalence and degree for Lipschitz continuous maps*, Arch. Rational Mech. Anal. **81** (1983), no. 3, 289–299. MR **84e**:49015
- [33] D. Ralph, *A chain rule for nonsmooth composite functions via minimisation*, Bull. Austral. Math. Soc. **49** (1994), no. 1, 129–137. MR **94m**:49026
- [34] D. Ralph and S. Scholtes, *Sensitivity analysis of composite piecewise smooth equations*, Math. Programming **76** (1997), no. 3, Ser. B, 593–612. MR **98c**:90153
- [35] D. Ralph and H. Xu, *Implicit smoothing and its application to optimization with piecewise smooth equality constraints*, J. Optim. Theory Appl. **124** (2005), no. 3, 673–699. MR **2006a**:90148
- [36] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR **43** #445
- [37] ———, *A property of piecewise smooth functions*, Comput. Optim. Appl. **25** (2003), no. 1–3, 247–250, A tribute to Elijah (Lucien) Polak. MR **2004d**:49040
- [38] S. Scholtes, *Introduction to Piecewise Differentiable Equations*, Habilitation thesis, University of Karlsruhe, Karlsruhe, Germany, 1994.

- [39] T. H. Sweetser, *A minimal set-valued strong derivative for vector-valued Lipschitz functions*, J. Optimization Theory Appl. **23** (1977), no. 4, 549–562. MR 57 #8012
- [40] L. Thibault, *Subdifferentials of compactly Lipschitzian vector-valued functions*, Ann. Mat. Pura Appl. (4) **125** (1980), 157–192. MR 82h:58006
- [41] ———, *On generalized differentials and subdifferentials of Lipschitz vector-valued functions*, Nonlinear Anal. **6** (1982), no. 10, 1037–1053. MR 85e:58020
- [42] J. Warga, *Derivative containers, inverse functions, and controllability*, Calculus of variations and control theory (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975; dedicated to Laurence Chisholm Young on the occasion of his 70th birthday), Academic Press, New York, 1976, pp. 13–45; errata, p. 46. Math. Res. Center, Univ. Wisconsin, Publ. No. 36. MR 55 #592
- [43] ———, *Fat homeomorphisms and unbounded derivate containers*, J. Math. Anal. Appl. **81** (1981), no. 2, 545–560, (Errata: *ibid.* **82** (1982), no. 2, 582–583). MR 83f:58007

INSTITUTE OF MATHEMATICS, UNIVERSITY OF DEBRECEN, H-4010 DEBRECEN, PF. 12, HUNGARY  
*E-mail address:* pales@math.klte.hu

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824, U.S.A.  
*E-mail address:* zeidan@math.msu.edu